MA 102

SHORT NOTES

1. Piece wise Continuous functions →

• A funⁿ f is piecewise Continuous on interval $[a, b]$

if $f(a^+)$ & $f(b^-)$ exist {limit exist}

f must be finite and Continuous at each interval.

* Imp. property → its boundedness and integrability over closed interval.

2. Piecewise Smooth functions →

f → $[a, b]$ be smooth if ~~Inter~~

f is piecewise Continuous on $[a, b]$

f' → exist & should be piecewise continuous $\lim_{x \to c^+} f'(x)$ → exist

$\lim_{x \to a^+} f'(x)$ & $\lim_{x \to b^-} f'(x)$ ⇒ exist.

3. Periodic functions →

$f(x) = f(x+T)$ ⇒ f is T periodic.

Properties of periodic function →

a. Sum, difference, product and quotient of two periodic function is are periodic.

a'). $H = f \pm g$ let $f \to T$ periodic, $g \to T'$ periodic

Ex $\cos x + \cos \pi x$ ↓ not periodic

⇒ H = COMMON of (f & g) ⇒ $\frac{\text{LCM (upper part)}}{\text{HCF (lower part)}}$

2). $f(x) \rightarrow T$ periodic

$f(ax) \rightarrow \frac{T}{a}$ periodic

$f(x/a) \rightarrow aT$ periodic

3) a constant no. is always periodic with infinite period.

4). $$\int_0^T f(x)dx = \int_a^{a+T} f(x)dx = \int_b^{b+T} f(x)dx$$

5). $g(f(x+T)) = g(f(x)) \rightarrow T$ periodic if $f \,\&\, g \Rightarrow T$ periodic
both
or g is not periodic

4). Trigonometric polynomials and Series →

Trigonometric polynomial of order n

$$S_n(x) = a_0 + \sum_{n=1}^{n} \left(a_k \cos\frac{\pi k x}{l} + b_k \sin\frac{\pi k x}{l}\right)$$

period of $S_n(x) = \frac{2\pi}{\pi/l} = 2l$ {common}.

- Infinite trigonometric Series →

$$S(x) = a_0 + \sum_{n=1}^{\infty} \left(a_k \cos\frac{\pi k x}{l} + b_k \sin\frac{\pi k x}{l}\right)$$

- ~~iff~~ if it is converges, then it represent a function of period $2l$.

← Results to remember →

- $\int_0^T \sin(k\omega_0 t)\,dt = \int_0^T \cos(k\omega_0 t)\,dt = 0$

$\int_0^T \sin^2(k\omega_0 t)\,dt = \int_0^T \cos^2(k\omega_0 t)\,dt = T/2$ $\left\{ \begin{matrix} \text{if } T = 2\pi^* \\ T/2 = \pi \end{matrix} \right\}$

5. orthogonality property of trigonometric system →

inner product → $\langle f(x)\, g(x) \rangle = \int_a^b f(x) g(x)\,dx$

if inner product $= 0$ then $f(x)$ & $g(x)$ are called orthogonal in interval $[a,b]$

& written as :→ $f \cdot g \in C_p^0\, [a,b]$

- basic trigonometric system

$1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x \ldots$

is orthogonal on interval $[-\pi, \pi]$ or $[0, 2\pi]$.

$\int_0^T \cos(k\omega_0 t) \sin(g\omega_0 t)\,dt = 0$

$\int_0^T \sin(k\omega_0 t) \sin(g\omega_0 t)\,dt = 0$

$\int_0^T \cos(k\omega_0 t) \cos(g\omega_0 t)\,dt = 0$

NOTE →

The value of the integral over length of periodic of integrand is equal to zero if the integrand is a product of two different members of trigonometric system.

6. fourier series →

Gen. form.

$$f = \frac{a_0}{2} + \sum_{K=1}^{\infty} a_K \cos(Kx) + b_K \sin(Kx).$$

$$a_K = \frac{1}{\pi}\int_a^b f(x) \cos\left(\frac{2n\pi}{b-a}\right)x\,dx$$

$$b_K = \frac{1}{\pi}\int_a^b f(x) \sin\left(\frac{2n\pi}{b-a}\right)x\,dx$$

let f is 2π periodic

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx \quad , \quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx \qquad n = 1, 2, 3 \ldots$$

or

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

The exponential fourier integral / complex fourier integral ......|.......|.|.... 3.

fourier integral representation → 1

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(u) \cos\alpha(u-x)\, du\, d\alpha$$

or

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(\alpha)\cos\alpha x \cancel{d\alpha} + B(\alpha)\sin\alpha x] \cdot d\alpha$$

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t) \cos\alpha t \cdot dt$$

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t) \sin\alpha t \cdot dt$$

fourier Cosine Transformation →

$$f(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(u) \cos\alpha u \cos\alpha x\, du\, d\alpha$$

or

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\int_0^\infty f(u)\cos\alpha u\, du\right)\cos\alpha x\, d\alpha$$

Here

$$\hat{f}_c(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cos\alpha u\, du$$

FCT

then.

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(\alpha)\cos\alpha x\, d\alpha$$

Fourier sine transformation →

Fourier sine integral rep.

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(u) \sin\alpha u \, du \, \sin\alpha x \, d\alpha$$

or

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\int_0^\infty f(u) \sin\alpha u \, du \right) \sin\alpha x \, d\alpha$$

Here,

$$\hat{f}_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \sin\alpha u \, du$$

then.

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\alpha) \sin\alpha x \, d\alpha.$$

Properties of sine and cosine transformation →

1. $F_c(af + bg) = aF_c(f) + bF_c(g)$ → linearity

2. Transform of derivatives →

Cond" $f'(x)$ → Piecewise Continuous.

& $f(x) \to 0$, as $x \to \infty$

$F_s\{f'(x)\} = -\alpha F_c\{f(x)\}$ & $F_c\{f''(x)\} = -\alpha^2 F_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$

$F_c\{f'(x)\} = \alpha F_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0)$ & $F_s\{f''(x)\} = -\alpha^2 F_s\{f(x)\} - \sqrt{2/\pi}\, \alpha f(0)$

The exponential fourier integral / complex fourier integral representation.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\alpha(u-x)\,du\,d\alpha \rightarrow \text{even fun}^n \text{ of } \alpha$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\sin\alpha(u-x)\,du\,d\alpha = 0 \rightarrow \text{odd fun}^n \text{ of } \alpha$$

therefore

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\left[\cos\alpha(u-x) \pm i\sin\alpha(u-x)\right]du\,d\alpha$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\,e^{-i\alpha(u-x)}\,du\,d\alpha \leftarrow \text{inverse of}$$

or

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\,e^{i\alpha(u-x)}\,du\,d\alpha \leftarrow \text{complex fourier integral rep.}$$

Now

fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{i\alpha u}\,du\right]e^{-i\alpha x}\,.d\alpha$$

Here

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{i\alpha u}\,du \leftarrow \text{fourier transform of } f$$

&

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-i\alpha u}\,du \leftarrow \text{inverse fourier transform}$$

Properties of fourier transform →

1. linearity.

2. change of scale

if $F[f(x)] = \hat{f}(\alpha)$,

then $F(f(ax)] = \frac{1}{|a|} \hat{f}\left(\frac{\alpha}{a}\right)$.

3. shifting property →

$$F[f(x-a)] = e^{ia\alpha} F[f(x)] = e^{i\alpha a} \hat{f}(\alpha)$$

4. Duality property →

$$F(\hat{f}(x)] = f(-\alpha)$$

fourier transform of derivatives →

$f(x)$ continuously differentiable & $f(x) \to 0$ as $|x| \to \infty$

then.

$$F(f'(x)) = (-i\alpha) F(f(x)] = (-i\alpha) \hat{f}(\alpha)$$

for n^{th} derivative

$$F(f^n(x)] = (-i\alpha)^n F(f(x)] = (-i\alpha)^n \hat{f}(\alpha).$$

7. orthonormal →

$$||f|| = \left(\int_a^b [f(x)]^2 dx \right)^{1/2} = 1.$$

fourier representation theorem →

$$\frac{f(x^+) + f(x^-)}{2} = f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

converges to

NOTE → if a function is piecewise continuous.

then.

1). it is sufficient cond" for existence of fourier series.

2). it is possible to calculate fourier cofficients.

3). periodicity of a function is not required for developing fourier series.

8. Evalution of fourier cofficients for even and odd function

if f is even function.

then. $b_n = 0$ $\quad a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$

$n = 0, 1, 2.$

if f is odd function

then $a_n = 0$ $\quad b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

$a_0 = 0$

$n = 1, 2 \ldots$

6. 4.

9. Half Range fourier Series → $a_n = \frac{2}{l}\int_0^l a_n \cos\left(\frac{n\pi x}{l}\right) dx$

f(x) can be extending an even or an odd fun" of [-L, 0] or [0, L]

{cosine, sine or half range are exactly same for a & L course

cosine series

$$a_n = \frac{2}{\cancel{b-a}}\int_a^{\cancel{b}}$$

↓ Gen.

$$\cos\left(\frac{2n\pi x}{b-a}\right)$$

for Half

$$\left\{\cos\left(\frac{2n\pi x}{2(b-a)/2}\right)\right\}$$

10. fourier Integral representation of a function →

$$f(x) = \int_0^\infty F(\alpha)\, d\alpha = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(u)\cos\alpha(u-x)\, du\, d\alpha$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\left(\int_{-\infty}^\infty f(u)\cos\alpha u\, du\right)\cos\alpha x + \left(\int_{-\infty}^\infty f(u)\sin\alpha u\, du\right)\sin\alpha x\right] d\alpha$$

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]\, d\alpha$$

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\cos\alpha u\, du \quad \& \quad B(\alpha) = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\sin\alpha u\, du$$

* fourier integral representation

- Fourier Cosine integral representation.

$$f(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(u)\cos\alpha u\,du \cdot \cos\alpha x\,d\alpha$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\int_0^\infty f(u)\cos\alpha u\,du\right)\cos\alpha x\,d\alpha$$

fourier cosine transform of f in $0 < x < \infty$

$$\hat{f}_c(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cos\alpha u\,du$$

Notation $F_c(f)$

Inverse

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(x)\cos\alpha x\,d\alpha.$$

fourier sine integral representation →

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\int_0^\infty f(u)\sin\alpha u\,du\right)\sin\alpha x\,d\alpha$$

$$\hat{f}_s(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\sin\alpha u\,du$$

inverse →

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(\alpha)\sin\alpha x\,d\alpha$$

inverse fourier sine transform of $\hat{f}_s(\alpha)$

Notation $F_s^{-1}(\hat{f}_s)$

Important properties →

9. 1. linearity → let f & g piecewise Continuous and absolutely integrable fun"

$$F_c(af+bg) = aF_c(f) + bF_c(g)$$

$$F_s(af+bg) = aF_s(f) + bF_s(g)$$

2. Transform of derivaties →

let $f(x)$ be Continuous and absolutely integrable on X-axis.

$f'(x)$ piecewise Continous & $f(x) \to 0$ as $x \to \infty$

$$F_c\{f'(x)\} = \alpha F_s\{f(x)\} - \sqrt{\frac{2}{\pi}}\, f(0)$$

$$F_s\{f'(x)\} = -\alpha F_c\{f(x)\}$$

$$F_c\{f''(x)\} = -\alpha^2 F_c\{f(x)\} - \sqrt{\frac{2}{\pi}}\, f'(0)$$

$$F_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}\,\alpha f(0) - \alpha^2 F_s\{f(x)\}$$

Here we assumed that f & f' both are piecewise Continuous f & $f' \to 0$ as $x \to \infty$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\alpha(u-x)\,du\,d\alpha$$

$\rightarrow$ even fun" of α

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\sin\alpha(u-x)\,du\,d\alpha = 0$$

odd fun" of α

Combine $\Rightarrow$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\left[\cos\alpha(u-x) \pm i\sin\alpha(u-x)\right]du\,d\alpha$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{i\alpha(u-x)}\,du\,d\alpha$$

or

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{-i\alpha(u-x)}\,du\,d\alpha.$$

fourier transform of f

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{i\alpha u}\,du = F(f)$$

Inverse transform of f

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\alpha)\, e^{-i\alpha u}\,d\alpha : F^{-1}(f)$$

or $f(x) = \int_{-\infty}^{\infty} \hat{f}(\alpha)\, e^{i\alpha x}\, d\alpha$, $\hat{f}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u}\, du$

properties of fourier transform

1. linearity →

$$F(af + bg) = aF(f) + bF(g)$$

2. change of scale property →

$$F[f(ax)] = \frac{1}{|a|} \hat{f}\left(\frac{\alpha}{a}\right) , a \neq 0$$

3. shifting property →

$$F[f(x-a)] = e^{i\alpha a} F(f(x))$$

4. Duality property →

$$F[\hat{f}(x)] = f(-\alpha)$$

5. fourier transform of derivatives →

$f(x)$ → continuously differentiable

$f(x) \to 0$ as $|x| \to \infty$

$$F[f'(x)] = (-i\alpha) F(f(x)) = (-i\alpha) \hat{f}(\alpha)$$

$$F[f^n(x)] = (-i\alpha)^n F(f(x)) = (-i\alpha)^n \hat{f}(\alpha).$$